

# The halo mass function through the cosmic ages

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## ABSTRACT

In this paper we investigate how the halo mass function evolves with redshift, based on a suite of very large (with  $N_p = 3072^3 - 6000^3$  particles) cosmological N-body simulations. Our halo catalogue data spans a redshift range of  $z = 0 - 30$ , allowing us to probe the mass function from the dark ages to the present. We utilise both the Friends-of-Friends (FOF) and Spherical Overdensity (SO) halofinding methods to directly compare the mass function derived using these commonly used halo definitions. The mass function from SO haloes exhibits a clear evolution with redshift, especially during the recent era of dark energy dominance ( $z < 1$ ). We provide a redshift-parameterised fit for the SO mass function valid for the entire redshift range to within  $\sim 20\%$  as well as a scheme to calculate the mass function for haloes with arbitrary overdensities. The FOF mass function displays a weaker evolution with redshift. We provide a ‘universal’ fit for the FOF mass function, fitted to data across the entire redshift range simultaneously, and observe redshift evolution in our data versus this fit. The relative evolution of the mass functions derived via the two methods is compared and we find that the mass functions most closely match at  $z = 0$ . The disparity at  $z = 0$  between the FOF and SO mass functions resides in their high mass tails where the collapsed fraction of mass in SO haloes is  $\sim 80\%$  of that in FOF haloes. This difference grows with redshift so that, by  $z > 20$ , the SO algorithm finds a  $\sim 50 - 80\%$  lower collapsed fraction in high mass haloes than does the FOF algorithm, due in part to the significant over-linking effects known to affect the FOF method.

**Key words:** Large-scale structure of the universe: cosmology:theory—dark matter—galaxies:haloes—galaxies:high-redshift—methods: numerical

## 1 INTRODUCTION

The abundance of dark matter haloes – the halo mass function – plays an important role in cosmology due to its sensitivity to a number of important parameters including the matter density of the universe,  $\Omega_m$ , the Hubble parameter,  $h$ , the spectral index of the primordial power spectrum,  $n_s$ , and the dark energy equation of state (e.g. Holder et al. 2001; Haiman et al. 2001; Weller et al. 2002). Whilst the mass function is currently difficult to pin down with much precision observationally (Eke et al. 2006), it is relatively easy to probe via N-body cosmological simulations (see Lukic et al. (2007) for a review of older work, more recent studies include Tinker et al. (2008); Crocce et al. (2010); Courtin et al. (2011); Bhattacharya et al. (2011); Angulo et al. (2012)) and can, to an extent, be understood and modelled through analytic arguments (Press & Schechter 1974; Bond et al. 1991;

Lee & Shandarin 1998; Sheth & Tormen 2002; Maggiore & Riotto 2010; Corasaniti & Achitouv 2011; Lim & Lee 2012).

The halo mass function is employed widely in astronomy: at low redshifts it is used in statistical analyses of cluster surveys to constrain cosmological parameters (Smith & Marian 2011) as these large objects probe the high mass end of the function (for a recent review paper on the formation of galaxy clusters, including the role of the mass function, see Kravtsov & Borgani (2012)). It is a key component in studies employing correlation functions between galaxies, as the halo-halo term directly depends on the mass function (see for example Cooray & Sheth (2002) and references therein). At higher redshifts it is applied in modelling the process of reionisation, which proceeded between  $z \sim 6 - 20$  from sources residing in dark matter haloes including population-III stars, early galaxies and accreting black holes. Any significant observed deviation from the mass function predicted by the  $\Lambda$ CDM cosmological model would create tension in our current understanding of structure formation in the universe. For example we are able to put bounds on how massive the largest observable clusters in the visible

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universe should be from the mass function (see Harrison & Coles 2012, and references therein), the discovery of larger clusters would require explanation.

Much has been written on the topic of the mass function by groups working with N-body simulations. To date this has largely focussed on low redshift ( $z \lesssim 2$ ) studies – due to the difficulty in simulating haloes at high redshifts with any great accuracy – and haloes derived using the Friends-of-Friends (FOF) algorithm (Davis et al. 1985). There are notable exceptions, for example Reed et al. (2007) focuses on the FOF mass function at high redshifts ( $z > 10$ ) and Tinker et al. (2008) give a precision mass function for haloes derived using the Spherical Overdensity (SO) algorithm (Lacey & Cole 1994). There is currently no robust mass function available for haloes derived using the SO algorithm at redshifts higher than  $z \sim 2$  which represents a significant gap in the literature that we address in this work.

The majority of mass function studies performed to date in simulations have employed the FOF algorithm, not least because it is easier to implement and often computationally cheaper than the SO algorithm. Despite this, Tinker et al. (2008) and Kravtsov & Borgani (2012) argue that an SO halo definition is much more appropriate for correlating halo masses to X-ray, Sunyaev-Zeldovich (SZ) effect and optical observables. One of the purposes of this paper is to provide a useful SO mass function, calibrated to a  $\Lambda$ CDM cosmology, that covers as broad a redshift range as possible. We also calculate a universal FOF mass function and investigate how our data evolves with redshift against it.

The question of whether the mass function is ‘universal’ – independent of redshift or cosmology, when expressed in suitable units – has been answered recently in the negative by various authors (Reed et al. 2007; Tinker et al. 2008; Crocce et al. 2010; Courtin et al. 2011; Bhattacharya et al. 2011) despite there being studies prior to these that found little evidence for departures from universality (Jenkins et al. 2001; Lukic et al. 2007). It was originally noted by Jenkins et al. (2001) that in order to produce a universal mass function from FOF haloes one needs to take a constant linking length – the key parameter that the FOF algorithm uses, see § 3 – across all redshifts or cosmologies being considered. How appropriate this is is questionable as it has been observed that an important factor in creating FOF-based halo catalogues is the suppression of a systematic effect inherent to the algorithm known as ‘overlinking’ which occurs when two haloes are linked together by a bridge of particles. When this occurs the resultant objects are in no way physically interpretable as virialised collapsed density peaks and yet are almost always unavoidably included in FOF mass functions. Previous authors (Davis et al. 1985; Ma & Bertschinger 1995; Cole & Lacey 1996; Lukic et al. 2009) have calculated that 15-20% of all FOF haloes calculated with the standard linking length of 0.2 are objects that have been bridged together in this manner and that larger linking lengths lead to more overlinking. It is not clear to what extent overlinking impacts on the universality of the FOF mass function. Is the function closer to universal because the FOF haloes are overlinked in  $\sim 20\%$  of cases, or does the overlinking have a negative impact on the universal result? Studies by Courtin et al. (2011) and More et al. (2011) suggest that a suitable choice for the linking length parameter leads to a universal mass function and that the effect of overlinking is to distort this result.

Tinker et al. (2008) observed a much clearer departure from universality for SO haloes than for FOF haloes, a fact that More et al. (2011) propose is due to the effect of taking a fixed overdensity criterion – the key parameter of the SO algorithm, see § 3 – rather than specific overdensity criteria bespoke to a given

redshift or cosmology. One is therefore left with a choice: either adjust one’s halo definitions to attempt to produce a universal mass function or keep one’s halo definitions fixed and expect the function to deviate from a universal form. In this work we choose the latter. Deciding upon a suitable ‘universal’ redshift-dependent halo overdensity or linking length parameter is still an open question (although Courtin et al. (2011) and More et al. (2011) go some way towards providing a solution), due to complicating factors such as the triaxiality of haloes in the SO case – a purely spherical halo is an ideal that is never realised – and overlinking in the FOF case. We shy away from demanding a universal mass function and pin our halo definitions down firmly in § 3.

This paper is laid out as follows. In § 2 we describe the simulations we have used to construct the mass function. In § 3 we outline the halo-finding methods we employed and the definitions of what we are referring to by the term ‘halo’. In § 4 we present our mass function results and fits. In § 5 we present a summary of the work and a discussion of various points that arise.

## 2 SIMULATIONS

The simulations undertaken in this work are summarised in Table 1. They follow between  $3072^3$  (29 billion) to  $6000^3$  (216 billion) particles in a wide range of box sizes from  $11.4 h^{-1}\text{Mpc}$  up to  $6 h^{-1}\text{Gpc}$ . Spatial resolutions range from  $0.18 h^{-1}\text{kpc}$  to  $50 h^{-1}\text{kpc}$  while particle masses range from  $3.6 \times 10^3 h^{-1}\text{M}_\odot$  to  $7.5 \times 10^{10} h^{-1}\text{M}_\odot$ . This allows dark matter haloes to be resolved in a very large mass range, with a low end of  $7.3 \times 10^4 h^{-1}\text{M}_\odot$  (for a 20 particle halo) in our smallest volumes, and no upper limit in the halo mass, since our largest volumes approach the size of the observable universe. All simulations were performed using the CubeP<sup>3</sup>M N-body code (Harnois-Deraps et al. 2012). The CubeP<sup>3</sup>M code calculates short-range direct particle-particle forces combined with a long-range PM force calculation making it a P<sup>3</sup>M (particle-particle-particle-mesh) code. It is massively parallel and runs efficiently on either distributed- or shared-memory machines. This is achieved via a cubical, equal-volume domain decomposition combined with a hybrid OpenMP and MPI approach. CubeP<sup>3</sup>M scales well up to thousands of processors and to date has been run on up to 21,976 computing cores (Iliev et al. 2008, 2010; Harnois-Deraps et al. 2012). We note that such large simulations yield very large amounts of data, with just the particle data amounting to between 700 GB ( $3072^3$ ) and 4.7 TB ( $6000^3$ ) per time-slice, providing significant challenges in the data handling and analysis, as we discuss below.

### 2.1 Cosmology

We base our simulations on the 5-year WMAP results (Dunkley et al. 2009; Komatsu et al. 2009). With the exception of two of our runs (the  $1 h^{-1}\text{Gpc}$  and  $3.2 h^{-1}\text{Gpc}$  boxes) the cosmology used for the simulations was the ‘Union’ combination from Komatsu et al. (2009), based on results from WMAP, baryonic acoustic oscillations and high-redshift supernovae; i.e.  $\Omega_m = 0.27$ ,  $\Omega_\Lambda = 0.73$ ,  $h = 0.7$ ,  $\Omega_b = 0.044$ ,  $\sigma_8 = 0.8$ ,  $n_s = 0.96$ . The  $1 h^{-1}\text{Gpc}$  and  $3.2 h^{-1}\text{Gpc}$  boxes were based on the slightly different ‘Alternative’ combination from the same paper; i.e.  $\Omega_m = 0.279$ ;  $\Omega_\Lambda = 0.721$ ,  $h = 0.701$ ,  $\Omega_b = 0.046$ ,  $\sigma_8 = 0.817$ ,  $n_s = 0.96$ . The power spectrum and transfer function used for setting initial conditions was generated using CAMB (Lewis et al. 2000).

**Table 1.** N-body simulation parameters. Background cosmology is based on the WMAP 5-year results. Minimum halo mass and the redshift of formation of the first halo are based on a 20 particle minimum. Resolution here refers specifically to the softening length, above which the interparticle force is the exact Newtonian force (see Harnois-Deraps et al. (2012) for details). See text for details regarding the quantities  $\mathcal{P}(k)_{max}$  and  $\delta_{in}^{rms}/\Delta_p$ .

Box Size	$N_{part}$	Mesh	Resolution	$m_{particle}$	$M_{halo,min}$	$z_{in}$	$\mathcal{P}(k)_{max}$	$\delta_{in}^{rms}/\Delta_p$	$z_{firsthalo}$
$h^{-1}\text{Mpc}$			$h^{-1}\text{kpc}$	$h^{-1}\text{M}_{\odot}$	$h^{-1}\text{M}_{\odot}$				
11.4	3072 <sup>3</sup>	6144 <sup>3</sup>	0.18	$3.63 \times 10^3$	$7.28 \times 10^4$	300	$2.0 \times 10^{-5}$	0.037	41
20	5488 <sup>3</sup>	10976 <sup>3</sup>	0.18	$3.63 \times 10^3$	$7.28 \times 10^4$	300	$2.0 \times 10^{-5}$	0.045	44
114	3072 <sup>3</sup>	6144 <sup>3</sup>	1.86	$3.83 \times 10^6$	$7.63 \times 10^7$	300	$1.2 \times 10^{-5}$	0.069	30
425	5488 <sup>3</sup>	10976 <sup>3</sup>	3.87	$3.69 \times 10^7$	$7.35 \times 10^8$	300	$9.5 \times 10^{-6}$	0.068	25
1000	3456 <sup>3</sup>	6912 <sup>3</sup>	14.47	$1.96 \times 10^9$	$3.85 \times 10^{10}$	150	$4.6 \times 10^{-4}$	0.057	17
3200	4000 <sup>3</sup>	8000 <sup>3</sup>	40.00	$4.06 \times 10^{10}$	$8.12 \times 10^{11}$	120	$4.5 \times 10^{-4}$	0.052	11
6000	6000 <sup>3</sup>	12000 <sup>3</sup>	50.00	$7.49 \times 10^{10}$	$1.50 \times 10^{12}$	100	$2.8 \times 10^{-5}$	0.051	11

## 2.2 Initial Redshift and Initial Conditions

The CubeP<sup>3</sup>M code uses first-order Lagrangian perturbation theory (1LPT), i.e. the Zel’dovich approximation (Zel’dovich 1970), to place particles in their initial positions. The initial redshift when this step takes place is given for each simulation in table 1.

There is debate regarding the suitability of the Zel’dovich approximation in setting initial conditions (Crocce et al. 2006, 2010; Lukic et al. 2007; Tinker et al. 2008; Knebe et al. 2009; Reed et al. 2012). Care must be taken when setting a choice for the initial redshift so as to ensure that artefacts (transients) are minimised. A detailed study by Lukic et al. (2007) emphasises the need for a suitable choice for the initial redshift as applied to a particle-mesh (PM) code (MC<sup>2</sup> – see Heitmann et al. 2005; Heitmann et al. 2008) using 1LPT. Reed et al. (2012) recommend the use of second-order Lagrangian perturbation theory (2LPT) in setting initial conditions based on their study using two tree-based codes: Gadget-2 (Springel 2005) and PKDGRAV (Stadel 2001).

Lukic et al. (2007) proposed two criteria to guide the choice of initial redshift for simulations employing 1LPT: (1) whether the amplitude of the initial power spectrum modes in the box are in the linear regime, and (2) whether the initial particle displacement is small enough so that the particle grid distortion is relatively small. The first criterion sets a minimum for the initial redshift based on the requirement that the dimensionless power spectrum,  $\mathcal{P}(k) = k^3 P(k)/2\pi^2$ , be less than some arbitrary, small value at the initial redshift (Lukic et al. (2007) used  $\mathcal{P}(k)_{max} \leq 0.01$ ). As smaller boxes for a given particle number probe higher values of  $k$ , earlier initial redshifts are required for smaller boxes. Similarly, for a fixed box size, simulations with greater numbers of particles also require earlier initial redshifts. The second criterion places a more stringent bound on the initial redshift. The example discussed in Lukic et al. (2007) was that the particles were displaced, on average, by no more than  $\delta_{in}^{rms} = 0.3\Delta_p$ , where  $\Delta_p = L_{box}/n_p$  is the interparticle spacing and  $\delta_{in}^{rms}$  is the root-mean-squared displacement of the particles. In Table 1 the values of  $\mathcal{P}(k)_{max}$  and  $\delta_{in}^{rms}/\Delta_p$  for our simulations are given. We see that for all of our simulations both of these criteria are satisfied.

## 3 HALO FINDING

### 3.1 Motivation

In the hierarchical picture of structure formation peaks in the linear density field of the early universe grow into highly non-linear structures via gravitational attraction. A popular formalism for describing this process is the identification of the non-linear, high density peaks as dark matter haloes. These haloes, appropriately defined, are concomitant with virialised regions in the universe where galaxies reside. The hierarchical growth of structure in this picture can then be viewed as a series of halo mergers, commencing with very small haloes at early times, and continuing all the way to the present, with haloes growing larger via both mergers and the accretion of smaller haloes. In cosmological simulations it is usual to attempt to track the progress of the growth of structure by identifying haloes within the simulation volume across a range of redshifts.

### 3.2 Halo Finding Approaches

There are two main approaches to finding haloes in cosmological simulations. The first is the spherical overdensity (SO) method. In this method haloes are assumed to be spherical. The extent of a halo is governed by a free parameter, the overdensity criterion,  $\Delta$ , which is a cut-off in density with respect to some background density (typically either the matter background density or the critical density of the universe). In the SO algorithm spheres are grown from a central location until the enclosed overdensity of the sphere is equal to the overdensity criterion. The main differences between halofinding codes that use this approach lie in how the centres of the candidate haloes are identified.

The second approach is the Friends-of-Friends (FOF) algorithm. This algorithm is also based on one parameter,  $b$ , the linking length parameter. The algorithm finds haloes that contain particles that are within  $b\Delta_p$  of at least one other particle in the halo. While the SO algorithm produces a halo that is by definition spherical, the FOF algorithm creates haloes that are arbitrarily shaped. For a recent comparison project on halofinding codes see Knebe et al. (2011).

We employ three halo-finding codes in our analysis: CubeP<sup>3</sup>M’s own on-the-fly SO halofinder (hereafter ‘CPMSO’) (Harnois-Deraps et al. 2012), the Amiga Halo Finder (hereafter

‘AHF’) (Gill et al. 2004; Knollmann & Knebe 2009), and the FOF halofinder from the Gadget-3 N-Body cosmological code (an update to the publicly available Gadget-2 code (Springel 2005)).

The CPMSO halofinder utilises a fine mesh from the CubeP<sup>3</sup>M code (a mesh with spacing of  $\Delta_p/2$ ) to identify local peaks in the density field. The code first builds the fine-mesh density using either Cloud-In-Cell (CIC) or Nearest-Grid-Point (NGP) interpolation. It then proceeds to search for and record all local density maxima above a certain threshold (typically set to 100 above the mean density) within the physical volume. It then uses quadratic interpolation on the density field to determine more precisely the location of the maximum within the densest cell. The halo centre determined this way agrees closely with the centre-of-mass of the halo particles. Each of the halo candidates is inspected independently, starting with the highest peak. The grid mass is accumulated in spherical shells of fine grid cells surrounding the maximum until the mean overdensity within the halo drops below  $\Delta$ . While the mass is accumulated it is removed from the mesh, so that no mass element is double-counted. This method is thus inappropriate for finding sub-haloes as within this framework they are naturally incorporated in their host haloes. Because the haloes are found on a grid of finite-sized cells and spherical shells constructed from them, it is possible, especially for the low-mass haloes, to overshoot the target overdensity. When this occurs we use an analytical halo density profile to correct the halo mass and radius to the values corresponding to the target overdensity. This analytical density profile is given by the Truncated Isothermal Sphere (TIS) profile (Shapiro et al. 1999; Iliev & Shapiro 2001) for overdensities below  $\sim 130$ , and  $1/r^2$  for lower overdensities. The TIS density profile has a similar outer slope (the relevant one here) to the Navarro, Frenk and White (NFW) profile (Navarro et al. 1997), but extends to lower overdensities and matches well the virialization shock position given by the Bertschinger self-similar collapse solution (Bertschinger 1985). For further details on the CPMSO method see Harnois-Deraps et al. (2012).

Details for AHF can be found in Knollmann & Knebe (2009). The algorithm identifies density peaks using a nested set of grids that are finer-grained in regions of higher density. Haloes are then identified by collecting particles together that are contained within isodensity contours on each level of the grid hierarchy. This allows AHF to identify sub-haloes within host haloes as the algorithm progresses from high resolution grids to lower resolution ones. AHF then removes particles that are unbound and recalculates halo properties based on the remaining bound particles.

The specifics of the FOF halofinder packaged in with the Gadget-3 code currently have not been detailed in any publication but the algorithm itself is outlined in Davis et al. (1985). The main difference in the algorithm that exists in the Gadget-3 version is that the code is parallelised for distributed-memory machines. Specifically, haloes are found in local subvolumes of the simulation assigned to individual MPI tasks (created using the Gadget-3 domain decomposition which utilises a space-filling Peano-Hilbert curve – for details see the Gadget-2 paper Springel (2005)) and then haloes that extend spatially beyond the edges of the subvolumes are linked together in a final MPI communication step. We have altered the Gadget-3 code to read CubeP<sup>3</sup>M’s particle output format and significantly reduced its memory footprint by stripping away extraneous data structures.

Finally, we note that due to limitations in the scaling of the codes with processor numbers, large memory footprint, and incompatible data structures, in order to apply the AHF and FOF algorithms to our data it was necessary to split the simulation time-

slices into a number of subvolumes and run the halofinding algorithms on each subvolume independently. Each subvolume included a buffer zone which overlapped with the neighbouring ones, for correct handling of haloes straddling two or more sub-regions. We then stitched the subvolumes back together to create the final AHF and FOF halo catalogues, removing any duplicated structures in the overlapping buffers. Although somewhat more expensive than applying each halo finder directly on the full data, this approach allows the handling of much larger amounts of data than otherwise possible and provides additional flexibility in terms of computational resources needed for post-processing.

### 3.3 Halo Definitions

In this work we choose the overdensity criteria for identification of spherical overdensity haloes to be  $\Delta_{178}$ , i.e. an overdensity of 178 times the background matter density. This is a common choice motivated from the top-hat model of non-linear collapse in an Einstein de-Sitter (EdS) universe (Gunn & Gott 1972). The overdensity criterion is usually taken to be an overdensity with respect to either the background matter density or the background critical density. For an EdS universe, i.e. a universe containing only collisionless matter, the two are the same. For a  $\Lambda$ CDM universe, at late times, the two deviate from each other due to the increasing dark energy component that contributes to the critical density of the universe but not to the matter density. Care must be taken to compare like-for-like overdensities, especially during the  $\Lambda$ -dominated epoch (note that all references in this work to overdensities refer to those with respect to the background matter density). Any given non-linear overdensity criterion can be mapped onto a threshold value in the linear regime. For  $\Delta_{178}$  this corresponds to a linear overdensity of  $\delta_c = 1.686$  in an EdS universe. In a  $\Lambda$ CDM universe the value of  $\delta_c$  evolves to 1.674 at  $z = 0$  due to the influence of dark energy. In the case of the FOF halofinder we follow various previous authors (Jenkins et al. 2001; Reed et al. 2003, 2007; Crocce et al. 2010; Courtin et al. 2011; Angulo et al. 2012) and use a linking length of 0.2 for our analysis.

### 3.4 Halo Mass Redshift Dependence

There are a number of factors that influence the growth of halo mass. They can be divided into ‘physical’ mass growth, i.e. the growth of the haloes via mergers and accretion, and, following the nomenclature of Diemer et al. (2012), ‘pseudo’ mass growth. For a recent study on the relative effects of physical and pseudo mass growth on the SO mass function between  $z = 0 - 1$  see Diemer et al. (2012) and references therein. An important point from this study is that a significant fraction of SO halo mass growth between  $z = 0 - 1$  is due to pseudo-evolution of halo mass. This is growth of halo mass, defined in the SO sense, due to the evolution of the background density of the universe. As the background matter density of the universe decreases with the universe’s expansion, the radius of a given SO halo will grow as it requires a lesser enclosed physical density to meet the overdensity criterion of  $\Delta$  times the background density. In the FOF case there also exists a pseudo mass growth: as the linking length is defined as a constant length in co-moving coordinates its physical length increases as  $z$  decreases. The implication of this is that for a halo with a static profile the extent of an FOF halo will increase with time. Fakhouri & Ma (2010) split FOF halo growth into an accretion component and a diffuse component, the latter containing an element of pseudo growth. Cur-

rently there is no equivalent study to the Diemer et al. (2012) work on pseudo mass growth in SO haloes for FOF haloes.

In addition to the pseudo evolution of halo mass there is the evolution of  $\delta_c$  (and  $\Delta$ ) due to dark energy dominance at low redshifts. This has well-studied implications for the SO halo mass definition (Lahav et al. 1991; Lacey & Cole 1993; Eke et al. 1996). Because of this  $\delta_c(z)$  is often used as a parameter in mass functions (Press & Schechter 1974; Sheth & Tormen 2002; Reed et al. 2003, 2007; Courtin et al. 2011). In this work we provide an SO mass function fit for haloes in an  $\Lambda$ CDM universe that is based on  $\Delta_{178}$  and do not include  $\delta_c(z)$  as a variable in the fit. This is motivated by the fact that we parameterise redshift evolution occurring due to dark energy, pseudo mass growth, and systematic effects such as halo mergers (which occur at different rates in different epochs) and non-sphericity of haloes. These latter effects are not captured by the evolution of  $\delta_c(z)$ . We also note that a baryonic component will have an effect on the mass function (estimated by Cui et al. (2012) to increase the mass function by  $\lesssim 3\%$  for a  $\Delta_{200}$  halo definition) not captured in this work.

An issue that pseudo mass evolution raises is that any redshift evolution of the mass function (whether expressed in terms of mass or  $\ln\sigma^{-1}$ , see § 4) is due to an interplay between halo definition and the physical increase of halo mass with time. In order to link halo definitions to observations and compare our results to previous studies on the halo mass function we adopt the common halo definitions above and attempt to capture redshift evolution based on them.

#### 4 THE HALO MASS FUNCTION

The mass function can be expressed in a number of ways. Here we use the multiplicity function,  $f(\sigma, z)$ , which represents the fraction of mass that has collapsed to form haloes per unit interval in  $\ln\sigma^{-1}$ . This definition uses  $\sigma^{-1}$  as a proxy for mass, where  $\sigma^2$  is the variance of the linear density field, given by:

$$\sigma^2(M, z) = \frac{D^2(z)}{2\pi^2} \int_0^\infty k^2 P(k) W^2(k; M) dk \quad (1)$$

where  $P(k)$  is the power spectrum of the linear density field,  $W(k; M)$  is the Fourier-space representation of a real-space top-hat filter containing mass  $M$  – assuming that the top-hat sphere encloses a region that contains a mean density identical to that of the universe – and  $D(z)$  is the growth factor, normalised to the unity at  $z = 0$  (Peebles 1993). The radius in real-space of the filter can be set using an overdensity criterion,  $\Delta$ , as per the SO algorithm. The relationship between  $\sigma$  and halo mass for our two WMAP-5-based cosmologies is shown in Figure 1.

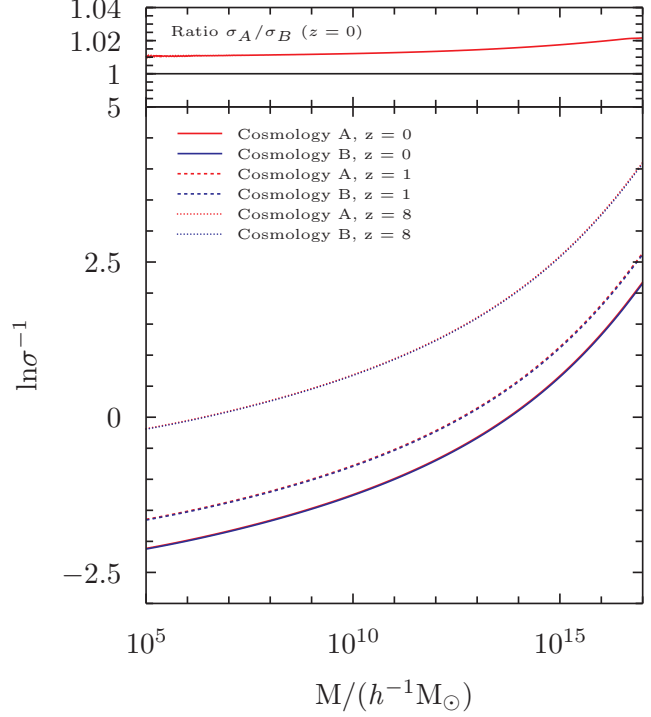
We define the halo multiplicity function as (Jenkins et al. 2001):

$$f(\sigma, z) \equiv \frac{M}{\rho_m(z)} \frac{dN(M, z)}{d\ln\sigma^{-1}} \quad (2)$$

where  $N(M, z)$  is the mass function proper, that is to say the number count of haloes with a mass less than  $M$  per unit volume, and  $\rho_m$  is the mean matter density of the universe.

##### 4.1 Mass Binning

The halofinders calculate values for the masses of all resolved haloes. It is then necessary, in order to construct  $f(\sigma, z)$ , to count the number of haloes in mass bins before converting the masses



**Figure 1.** The relationship between the variance of the linear density field,  $\sigma$ , and halo mass for two  $\Lambda$ CDM cosmologies. ‘Cosmology A’ refers to the ‘Union’ model of Komatsu et al. (2009) with  $\Omega_m = 0.27$ ,  $\Omega_\Lambda = 0.73$ ,  $h = 0.7$ ,  $\Omega_b = 0.044$ ,  $\sigma_8 = 0.8$ ,  $n_s = 0.96$ . ‘Cosmology B’ refers to the ‘Alternative’ model of Komatsu et al. (2009) with  $\Omega_m = 0.279$ ,  $\Omega_\Lambda = 0.721$ ,  $h = 0.701$ ,  $\Omega_b = 0.046$ ,  $\sigma_8 = 0.817$ ,  $n_s = 0.96$ .

into values of  $\ln\sigma^{-1}$ . Rather than equation 2, for simulated halo catalogues we have:

$$f(\sigma, z) \equiv \frac{M^2}{\rho_m(z)} \frac{\Delta N_{sim}}{\Delta M} \frac{d\log M}{d\ln\sigma^{-1}} \quad (3)$$

where care must be taken in choosing the width of the bins in mass as this can potentially result in a source of systematic error. For a detailed evaluation of this error see Lukic et al. (2007), who note that the error is not significant so long as the bin widths,  $\Delta\log M$ , do not exceed 0.5. We use bin widths for our analysis that remain constant in  $\log M$  with  $\Delta\log M = 0.16$ , and assign the mass of a given bin to the average of all the haloes in the bin rather than the bin centre.

##### 4.2 Error Treatment

Throughout this paper we use 1- $\sigma$  Poisson error bars as defined in Heinrich (2003):

$$\sigma_{\pm} = \sqrt{N_{haloes}} + \frac{1}{4} \pm \frac{1}{2} \quad (4)$$

The main advantages of using error bars in this form are: (1) for small numbers of haloes the asymmetry of the bars reflects the asymmetry of the Poisson distribution; (2) for a halo count of 1 the lower edge of the error bar does not reach zero. Note that for large numbers of haloes the errors tend to the expected  $\sqrt{N}$  form.

### 4.3 Data Treatment

In order to combine data across our simulations certain systematic effects need to be accounted for. These include: accounting for the finite volumes of our simulations; adjusting the FOF haloes for a systematic overestimation in mass for haloes sampled with low particle numbers; the question of whether to remove or include sub-haloes from our AHF data; and the choice of a lower limit to the number of particles a halo contains. The mass functions presented here are (1) corrected for finite volumes for simulations with a box size of less than  $425 h^{-1}\text{Mpc}$ , (2) adjusted, in the case of our FOF haloes, to account for mass overestimation using the empirical correction of Warren et al. (2006), (3) constructed based only on host haloes (in the case of data from the AHF halofinder the host haloes have masses that include any sub-haloes they contain) and, (4) have a minimum of 1000 particles in each halo. We now discuss each of these points in detail.

#### 4.3.1 Finite Volume Correction

The volume of space modelled by a cosmological simulation is always finite. However when we speak of the mass function we typically refer to a ‘global’ mass function, i.e. one that would correspond to an infinitely large volume. There is therefore a disconnect that needs to be bridged between simulations and an ideal, global, mass function (Sirko 2005; Bagla & Prasad 2006; Power & Knebe 2006). There exist a number of ways to address this issue. In this work we adopt an approach recently employed by Lukic et al. (2007) and Bhattacharya et al. (2011). We proceed by assuming that mass function universality holds strictly in the sense that the functional form is the same for both global and local volumes. This is similar to assuming that the mass function is universal across cosmologies that differ in their mass- $\sigma$  relations, as the effect of a finite box size is to set to zero the amplitude of any density fluctuations on a scale greater than the box size. There is evidence that the mass function is not universal (see § 5); that its functional form has a weak dependence on both redshift and cosmology. Therefore any finite volume correction we make based on the assumption of universality is an approximation. We adopt the approach here despite this, as it is relatively straight-forward to apply and it brings our data into better agreement across difference box sizes. We also briefly discuss below other possible methods that could be applied.

The approach is as follows. Assuming equation 2 refers to the mass function in an infinite simulation volume we can re-write it as:

$$\frac{dN}{dM} = \frac{\rho_m}{M^2} f(\sigma) \frac{d\ln\sigma^{-1}}{d\ln M} \quad (5)$$

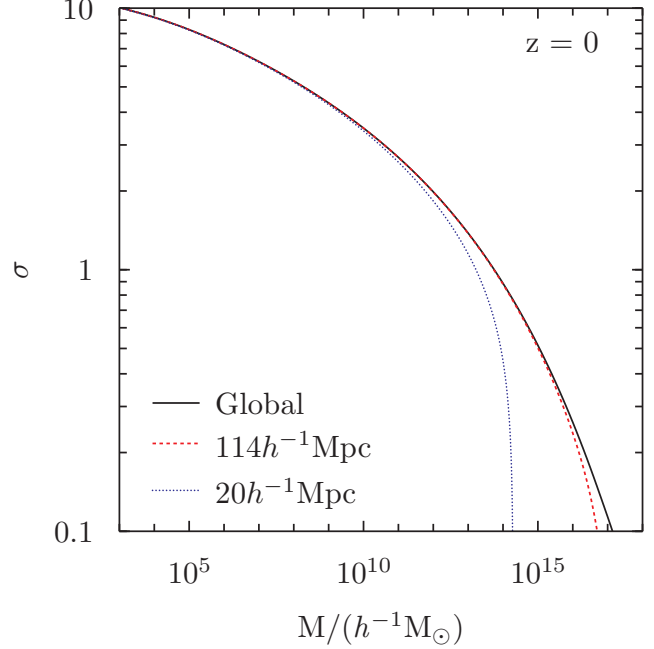
and now for a finite simulation volume we can write:

$$\frac{dN'}{dM'} = \frac{\rho_m}{M'^2} f(\sigma) \frac{d\ln\sigma'^{-1}}{d\ln M'} \quad (6)$$

where now  $\sigma'(M')$  is determined by the discrete power spectrum of the simulation in question. Also  $M'$  is a function of  $M$  defined such that  $\sigma(M) \equiv \sigma'(M'(M))$ . The assumption that the mass function is universal allows us to say that the multiplicity functions,  $f$ , in equations 5 and 6 are identical. This then leads to the relation:

$$dN = dN' \frac{dM'(M)}{dM} \quad (7)$$

We now require some method of connecting  $\sigma'(M')$  and  $\sigma(M)$ . The extended Press-Schechter formalism (Bond et al. 1991) approximately connects  $\sigma'(M')$  and  $\sigma(M)$  via:



**Figure 2.** The finite volume correction for small boxes. The global  $M - \sigma$  relation is adjusted to give a relation based on the volume of the simulation box in question via equation 8. The (approximate) assumption of mass function universality is then employed to adjust the masses and numbers of dark matter haloes via equations 7 and 8.

$$\sigma'(M')^2 = \sigma(M)^2 - \sigma_{R(box)}^2 \quad (8)$$

where  $\sigma_{R(box)}^2$  is the variance of fluctuations in spheres that contain the simulation volume. To align this to the simulations we make another approximation and equate the spherical volume to the cubical simulation volume. The resulting mass- $\sigma$  relations at  $z = 0$  for the  $114 h^{-1}\text{Mpc}$  and  $20 h^{-1}\text{Mpc}$  boxes are shown in Figure 2.

The steps in the volume correction are then as follows: (1) Calculate  $\sigma_{R(box)}^2$  for the simulation box in question. (2) For each bin in  $\sigma$  calculate an adjusted bin  $\sigma$  value,  $\sigma_{cor}$ , using  $\sigma_{cor}^2 = \sigma^2 - \sigma_{R(box)}^2$ . (3) Adjust each halo count using equation 7, where  $M$  is the mass that corresponds to  $\sigma$  and  $M'$  the mass that corresponds to  $\sigma_{cor}$ .

For comparative purposes we now discuss other solutions that have been employed to solve the finite volume issue. Reed et al. (2007) performed a number of different N-body simulations and calculated the range of  $\sigma$  for each one using the input power spectra of the simulations and the relationship between  $\sigma$  and mass for a finite box in the discrete case:

$$\sigma^2(M, z) = D^2(z) \sum_{\mathbf{k}} |\delta_{\mathbf{k}}|^2 W^2(k; M) \quad (9)$$

where  $|\delta_{\mathbf{k}}|$  is the linear amplitude of the Fourier modes in the simulation at  $z = 0$ . This approach has the advantage of both correcting for finite volumes and also compensating for cosmic variance. One drawback to it though is that each realisation has a different mass- $\sigma$  relationship. We did not adopt this approach as it is a method more suited to multiple simulation runs, where the mass- $\sigma$  relations can be averaged over to produce a mass function fit. We did not have the luxury of repeating our runs due to the large sizes of the simulations. Bagla et al. (2009) choose to make no specific correction (although they remove any data points that are affected by more than a threshold level of error – 10% in the number counts – from



their analysis in a similar manner to Tinker et al. (2008)). This decision was motivated by the observation that the mass function is an unknown function that is deduced from simulation data so to place any a priori constraints upon it – such as a universal functional form for global and local mass- $\sigma$  relations – is undesirable. This approach was applicable in the study undertaken by Bagla et al. (2009) as the box sizes used for their simulations were all relatively large ( $\geq 256 h^{-1}\text{Mpc}$ ), for our smaller boxes not correcting for finite volumes would lead to undesirable systematic discontinuities across our simulations. Yoshida et al. (2003) and Bagla & Prasad (2006) replaced equation 1 with:

$$\sigma_{box}^2(M, z) = \frac{D^2(z)}{2\pi^2} \int_{2\pi/L}^{\infty} k^2 P(k) W^2(k; M) dk \quad (10)$$

which takes a cut-off in low  $k$  modes at the size of the box. Lukic et al. (2007) note that a correction of this form has a dependence on the accuracy of the mass functions at redshifts greater than  $z = 5$ . They also note that in comparison to their correction method it exhibits offsets and shape changes across different box sizes. Finally, Barkana & Loeb (2004) utilised the extended Press-Schechter formalism to create a volume adjustment by taking the Sheth-Tormen mass function for a global volume and applying a correction to it to create a mass function suitable for a smaller volume. As we are not adopting a mass function in the Sheth-Tormen form this approach was inappropriate here.

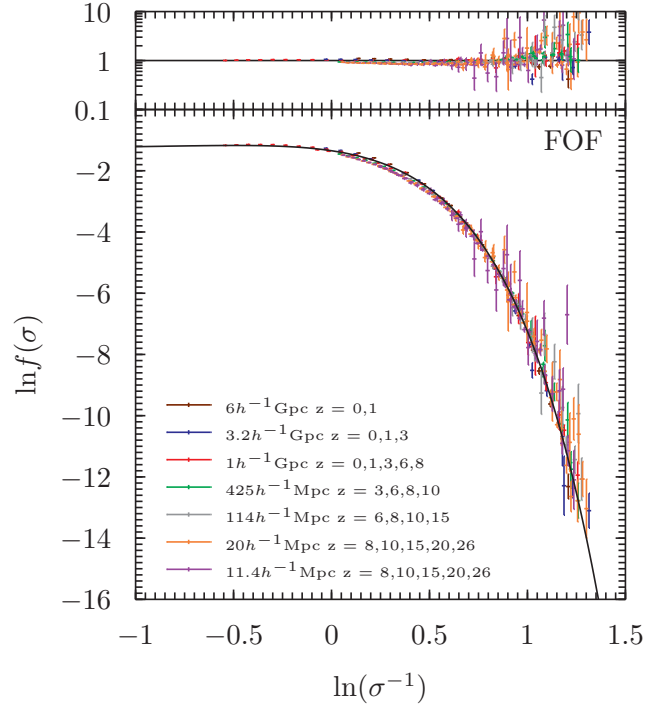
#### 4.3.2 Warren Correction to FOF Haloes

Warren et al. (2006) proposed a simple correction to the masses of FOF haloes that alleviates a systematic error in halo masses calculated via the FOF algorithm at low particle counts. This correction was devised based on analysis of FOF haloes at  $z = 0$ , but has been checked by Lukic et al. (2007) for FOF haloes at higher redshifts. The FOF algorithm overestimates the masses of haloes when there are low numbers of particles sampling the haloes. Warren et al. (2006) proposed the following correction to particle counts in haloes:

$$N_{corrected} = N(1 - N^{-0.6}) \quad (11)$$

We have adopted this correction in this work and all FOF data shown includes it. As we have also adopted a cut-off in particle number of 1000 the maximum effect this correction has on the masses of our FOF haloes is  $\sim 2\%$ .

There is debate regarding the appropriate correction to use. Bhattacharya et al. (2011) find that a correction of  $N_{cor} = N(1 - N^{-0.65})$  is more suitable and Lukic et al. (2009) note that the halo concentration parameter for haloes with an NFW profile also affects the correction, a result that is corroborated by More et al. (2011). The more sophisticated corrections proposed by Lukic et al. (2009) and More et al. (2011) (the former provide a correction for FOF haloes with  $b = 0.2$  whereas the latter provide a correction valid for different values of  $b$ ) are based on individual halo concentrations. We have not adopted this approach as we do not possess profile information for our individual FOF haloes. The difference between the Warren correction and the More et al. (2011) correction (which can be as much as 15% for haloes with lower particle counts) becomes slight for haloes with many particles. For our smallest, 1000 particle haloes, the correction to the Warren formula is of the order of  $< 5\%$  depending on halo concentrations. Not accounting for this therefore introduces a small systematic error which in the very worst case of 1000 particle haloes would alter the FOF masses by  $\sim 0.15\%$ . In addition More et al.



**Figure 3.** The FOF mass function across all simulations and redshifts along with our fit (equation 12). The top panel shows the ratio of the data to the fitting formula.

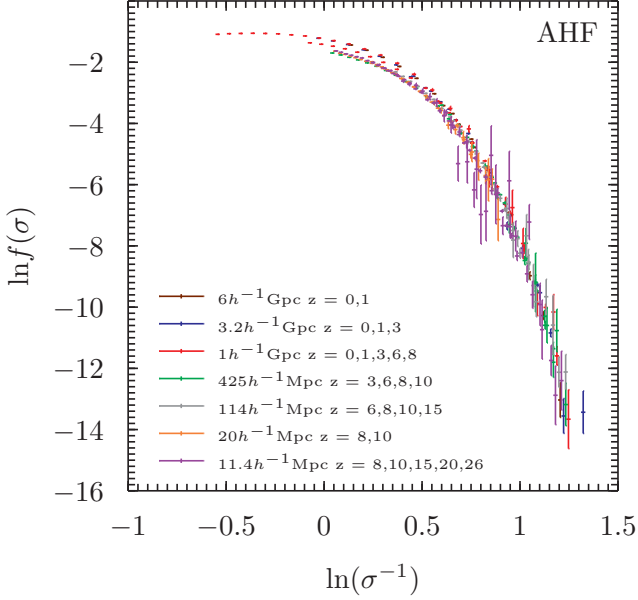
(2011) note that the Warren correction is specifically calculated to correct for the bias in the mass function itself whereas their study is one that corrects for the masses of individual FOF haloes when considered in isolation.

#### 4.3.3 Sub-Haloes

As noted in § 3 the AHF halofinder produces halo catalogues that contain sub-haloes – bound structures inside host haloes. For the purposes of constructing a mass function we have to make a choice as to how to deal with these haloes. The CPMSO and FOF halofinders both identify host haloes only, therefore for compatibility we make the natural choice of excluding the sub-haloes from the mass functions based on the AHF catalogue. The host haloes detected by AHF have properties, including mass, that are calculated based on all the bound matter contained within the halo, including the sub-haloes themselves. This results in the AHF host haloes being directly comparable to the CPMSO ones.

#### 4.3.4 Low-End Particle Cutoff

We take a minimum particle cut-off of 1000 particles per halo in this study. This is conservative as often halo-finders report haloes with as little as 20 particles, or even fewer. Lukic et al. (2007) recommend a 300 particle minimum for constructing a mass function with a pure PM code. Warren et al. (2006) take an ‘aggressive’ 400 particle minimum in their study and Tinker et al. (2008) a ‘conservative’ 400. It has been observed recently by Reed et al. (2012) that a minimum particle cut-off of at least 1000 is appropriate. Although this latter figure was quoted with tree-based N-body codes in mind – rather than a  $P^3M$  code – we use it to motivate our cut-off of 1000 particles. We note that for studies in other areas it might be



**Figure 4.** The AHF mass function across all simulations and redshifts.

acceptable to use haloes with lower particle counts, e.g. in clustering studies we need to reliably identify and locate a halo, but not necessarily know its mass or other properties precisely. For mass function calculations it is important that haloes are assigned masses that are as close to their correct values as possible, and more particles are required to ensure this is the case.

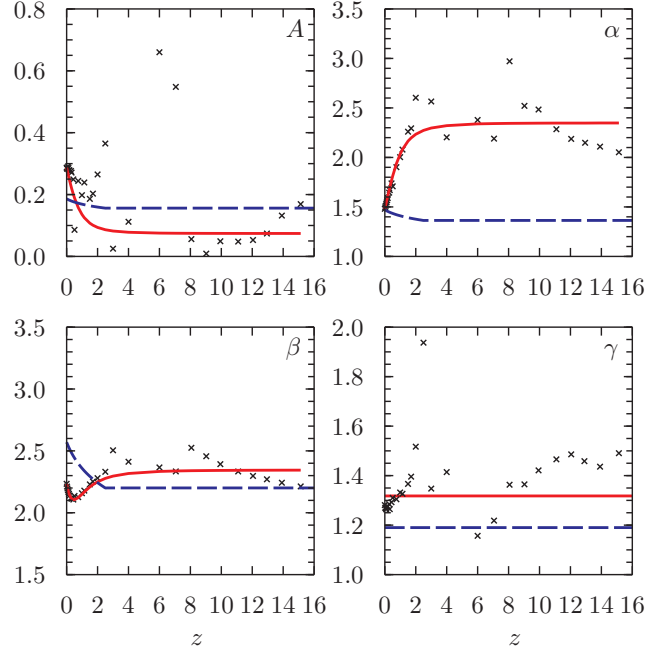
#### 4.4 The Universal FOF Function

We find that the halo mass function based on our FOF halo catalogues follows an universal fit applicable for data from all simulations across all redshifts. The fit is good to within  $\sim 10\%$  for most data points (Figure 3) and takes following the form:

$$f(\sigma) = A \left[ \left( \frac{\beta}{\sigma} \right)^\alpha + 1 \right] e^{-\gamma/\sigma^2} \quad (12)$$

where  $A = 0.282$ ,  $\alpha = 1.406$ ,  $\beta = 2.163$ ,  $\gamma = 1.210$ . This fit is valid in the range:  $-0.55 \leq \ln \sigma^{-1} < 1.31$ , which at  $z = 0$  corresponds to haloes with masses between  $1.8 \times 10^{12}$  and  $7.0 \times 10^{15} h^{-1} M_\odot$ . Our largest halo, found in the  $3.2 h^{-1} \text{Gpc}$  box at  $z = 0$  has a mass of  $6.4 \times 10^{15} h^{-1} M_\odot$  and  $\ln \sigma^{-1} = 1.17$ , whereas our highest  $\ln \sigma^{-1}$  value comes from a  $z = 3$  halo in the same simulation, with a mass of  $1.4 \times 10^{14} h^{-1} M_\odot$ . It should be noted that this extreme halo is most likely the result of the FOF algorithm linking together two large haloes via a bridge of particles – a systematic effect known as overlinking, discussed in § 5 below. At higher  $\ln \sigma^{-1}$  values, i.e. for high mass/high redshift, rare haloes, the scatter about the fit increases dramatically due to shot noise.

Whilst it is a remarkable result that over such a large range of  $z$  and  $\sigma$  we observe a halo mass function that approximately conforms to a universal shape we urge caution when using this fit for two reasons. Firstly the mass function is not completely universal and exhibits a modest redshift evolution (see § 5). Secondly it is difficult for a mass function derived from FOF haloes, that are arbitrarily shaped, to be related to observables (Kravtsov & Borgani 2012).



**Figure 5.** Evolution of the parameters in equation 12 for CPMSO haloes. The Tinker et al. (2008) values are shown in blue for comparison, the model used in this work is shown in red. The parameters have been fitted in the order:  $\gamma \rightarrow A \rightarrow \alpha \rightarrow \beta$ .

#### 4.5 Mass Function for Spherical Overdensity Haloes

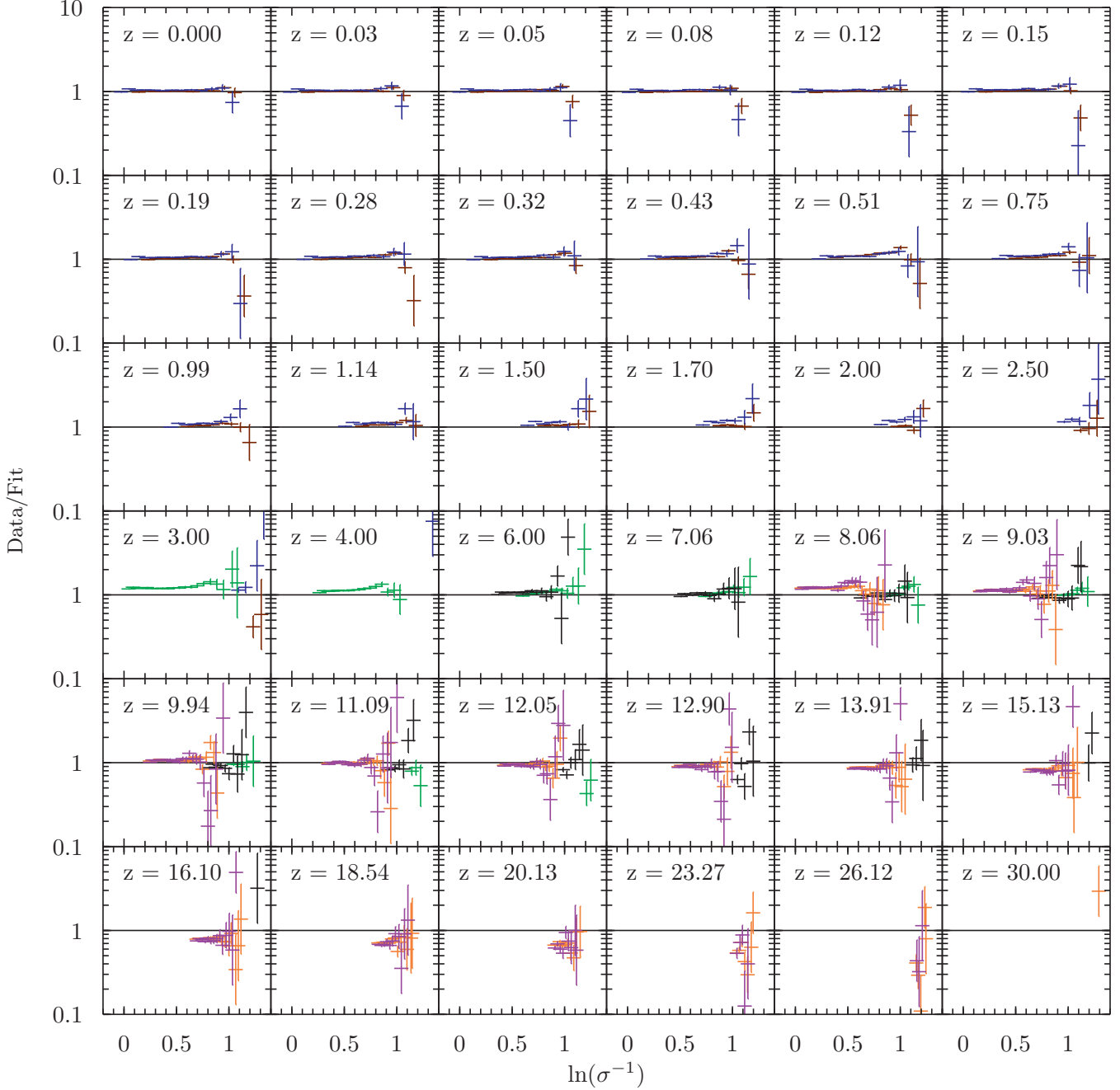
##### 4.5.1 Redshift Evolution

We show in Figure 4 the mass function from our AHF haloes. In contrast to the FOF mass function it is clear that a universal fit is not appropriate for spherical overdensity haloes (a fact also born out in the results from the CPMSO halofinder. For a comparison between the CPMSO and AHF halo mass functions see Figure 7 and discussion below). We again adopt the fitting function in equation 12, but with a parameterisation that includes a redshift-dependence, as discussed below. Since running AHF on multiple checkpoints from our largest simulations is computationally expensive we utilise the data from the CPMSO on-the-fly halofinder in order to investigate the redshift-dependence of the parameters for the spherical overdensity case. In § 4.5.2 below we discuss the differences between the three halo finding methods and re-scale the fit to match the AHF results. We perform a least-squares fit on the CPMSO data for 36 output redshifts, initially allowing all of the parameters in equation 12 to vary. Tinker et al. (2008) previously found, based on lower-redshift data ( $0 < z < 2$ ), that the parameter controlling the exponential cut-off scale,  $\gamma$ , is approximately constant across their range of redshifts. We similarly find that  $\gamma$  is approximately constant, albeit at a slightly different value from Tinker et al. (2008) (see Figure 5, bottom-right panel). Given this result we proceed by fixing this parameter at its value in the approximation of universality: i.e.  $\gamma = 1.318$ . We then fit the CPMSO data across all redshifts again. We find that using a parameterisation that includes  $\Omega_m(z)$ , the matter content of the universe at a given redshift, enables us to capture late time behaviour that differs from that at high-redshift. This gives us a fit for  $A(z)$  in the following form:

$$A(z) = \Omega_m(z) \{ 0.990 \times (1+z)^{-3.216} + 0.074 \} \quad (13)$$

With  $\gamma$  and  $A$  modelled we then repeat the procedure for  $\alpha$  and then  $\beta$ . We find for  $\alpha(z)$ :





**Figure 6.** Ratios between the CPMSO redshift-dependent mass function and the data from the CPMSO halofinder across 36 redshifts from  $z = 0 - 30$ . The colour scheme is as per Figure 3.

$$\alpha(z) = \Omega_m(z) \{ 3.136 \times (1+z)^{-3.058} + 2.349 \} \quad (14)$$

and for  $\beta(z)$ :

$$\beta(z) = \Omega_m(z) \{ 5.907 \times (1+z)^{-3.599} + 2.344 \} \quad (15)$$

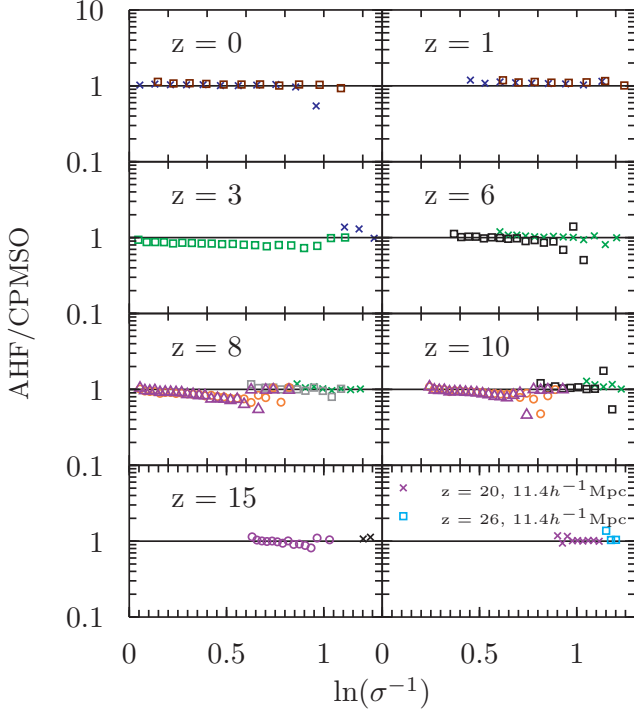
We show how our model compares to the parameter fitting data in Figure 5.

In Figure 6 we compare our redshift-dependent fit to our CPMSO data across all redshifts from  $z = 0$  and  $z = 30$ . We find that the fit is excellent at low redshifts and remains quite good, within 20%, all the way to  $z \sim 20$ . For  $z > 20$  our fit still gives a reasonable match, but the data has large error bars due to the scarcity of haloes then. Nonetheless, we note that around  $z = 3 - 4$  our

fit slightly under-predicts the abundances of SO haloes across the lower-mass  $\ln \sigma^{-1}$  range we cover. There is also an apparent under-prediction for lower  $\ln \sigma^{-1}$  around  $z = 8$  and there is perhaps an over-prediction at very high redshifts ( $z > 15$ ).

#### 4.5.2 AHF-based Fits

Our CPMSO on-the-fly halo finder is by its nature simplified and potentially more approximate than AHF. In Figure 7 we show the ratio between the mass functions derived using these two SO-based halofinders, AHF and CPMSO. In Figure 8 we show the ratio between the FOF mass function and the AHF mass function. Both



**Figure 7.** Comparison of the mass functions from the CPMSO and AHF halo finders across redshifts from  $z = 0 - 26$ . The colour scheme matches the simulations as per Figure 3

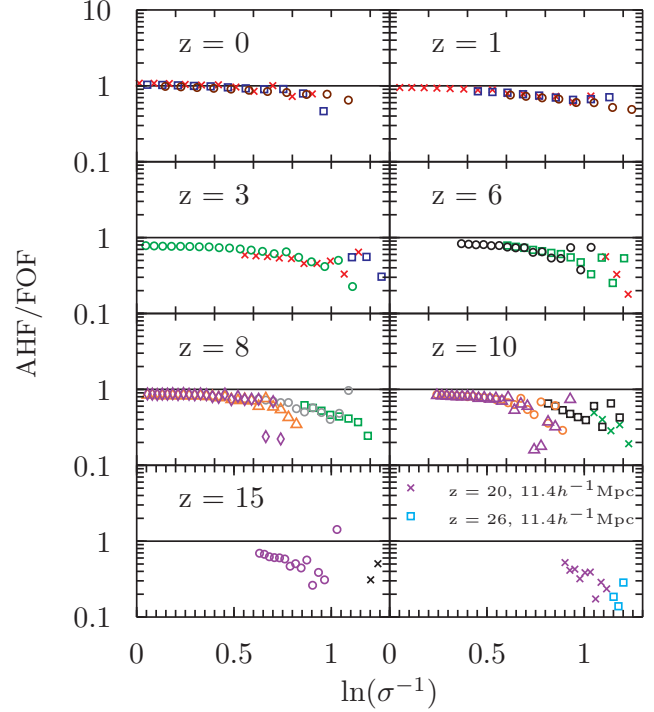
AHF and FOF exhibit consistent behaviour across all simulation volumes and resolutions and for all redshifts. The CPMSO halo finder largely agrees fairly well with AHF, typically within 10-20%, and much less at low redshifts. There are some systematic differences around  $z = 8$  in the transition between the  $20 h^{-1}\text{Mpc}$  and  $114 h^{-1}\text{Mpc}$  boxes. On the other hand, while both FOF and AHF are consistent across all box sizes, there is a systematic trend for the FOF to yield more rare, massive haloes than AHF for all volumes and redshifts. A similar trend was noted previously by Reed et al. (2007) and Tinker et al. (2008). Based on these results, we conclude that the AHF (and FOF) data is more consistent across different box sizes than the CPMSO halo finder.

Given these differences between the two SO halo finders, in addition to the CPMSO fit above, we also provide AHF fits to the data, with three different types: 1) a redshift-dependent one, based around the redshift evolution we have observed in the CPMSO mass function, but re-normalized to the AHF data; 2) a present-day one, based on our results at  $z = 0$ , for direct comparison with previous works; and 3) a more precise high-redshift, “Epoch of Reionisation (EoR)” halo mass function, based on all the AHF data past  $z = 6$ .

Given the relationship between the two SO halo finders in Figure 7 we expect that the redshift parameterisation presented in equations 13–15 applied to AHF data leads to a mass function that is roughly correct. The fit to AHF at  $z = 0$  can be improved by introducing a slightly different parameterisation for  $A$ , as follows:

$$A(z) = \Omega_m(z) \{1.097 \times (1+z)^{-3.216} + 0.074\} \quad (16)$$

The combination of equations 14, 15, 16, and  $\gamma = 1.318$  results in the ratios shown in Figure 9. We see that this fit is accurate to  $\sim 10\%$  for redshifts less than  $z = 15$ . The shape of the function in the low  $\ln \sigma^{-1}$  range is slightly different for  $z > 8$ , although



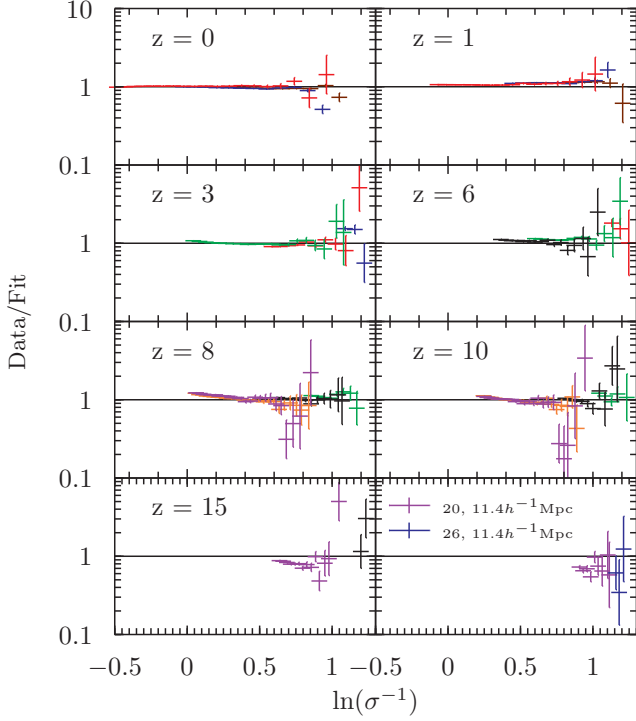
**Figure 8.** Comparison of the mass functions from the FOF and AHF halo finders across redshifts from  $z = 0 - 26$ . The colour scheme matches the simulations as per Figure 3

the amplitude and the shape are both correct for  $\ln \sigma^{-1} > 0.3$  at  $z = 8 - 10$ .

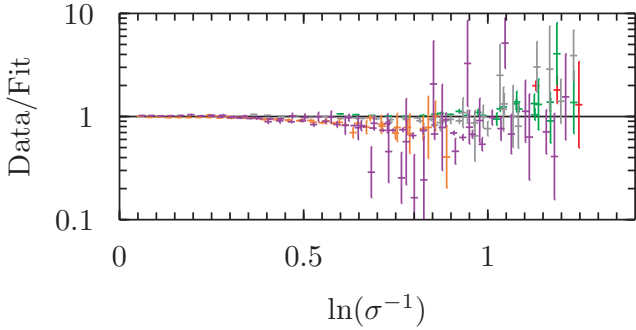
We now provide more accurate parameterisations for our AHF data for several different redshift ranges. At  $z = 0$  we have:  $A = 0.194$ ,  $\alpha = 2.267$ ,  $\beta = 1.805$  and  $\gamma = 1.287$  valid in the range  $-0.55 \leq \ln \sigma^{-1} < 1.05$ . As the redshift evolution of the spherical overdensity mass function is mainly apparent at later times ( $z < 3$ ) we can provide an universal fit that is appropriate for high redshift studies, for example for probing the EoR and the Cosmic Dark Ages. An AHF mass function fitted to all our data from  $z = 6$  upwards has the following parameterisation:  $A = 0.563$ ,  $\alpha = 0.874$ ,  $\beta = 3.810$  and  $\gamma = 1.453$  valid in the range  $-0.06 \leq \ln \sigma^{-1} < 1.24$  (corresponding at redshift  $z = 8$  to a mass range of  $3.5 \times 10^6 - 6.3 \times 10^{11} h^{-1} \text{M}_\odot$ ). These values are notably different to those given by our  $z = 0$  redshift parameterisation, due to the lack of constraining data for low values of  $\ln \sigma^{-1}$ . The ratio of the EoR fit versus our data is shown in Figure 10. There is considerable scatter in the high  $\ln \sigma^{-1}$  end due to data being incorporated from high redshifts when haloes are scarce. For the range  $0.05 \leq \ln \sigma^{-1} < 0.5$  the fit is accurate to within 10%.

#### 4.6 Varying $\Delta$

For practical purposes it is useful to have a simple method for adjusting the spherical overdensity mass function fits derived above to account for haloes defined with different overdensities. To this end we have run AHF on the  $1 h^{-1}\text{Gpc}$  box using values for  $\Delta$  ranging between 100 to 1600, at  $z = 0, 1$  and 3. The data has then been fitted with  $\Delta$  incorporated into the parameterisation. This allows, to good precision at lower redshifts, a mass function based on a given  $\Delta$  to be inferred from our  $\Delta_{178}$  fits. The results from this procedure are shown in Figure 11.



**Figure 9.** Ratios between the amplitude-modified CPMSO mass function and AHF data from  $z = 0$  to  $z = 26$ . The colour scheme is as per Figure 3.



**Figure 10.** Ratio between the EoR AHF fit and simulation data from  $z = 6$  to  $z = 26$ . The colour scheme is as per Figure 3.

The parameterisation is as follows. We assume that the  $f_{\Delta=178}$  mass function is suitable for describing mass functions with different choices for  $\Delta$  when it is suitably scaled by a function,  $\Gamma(\Delta, \sigma, z)$ :

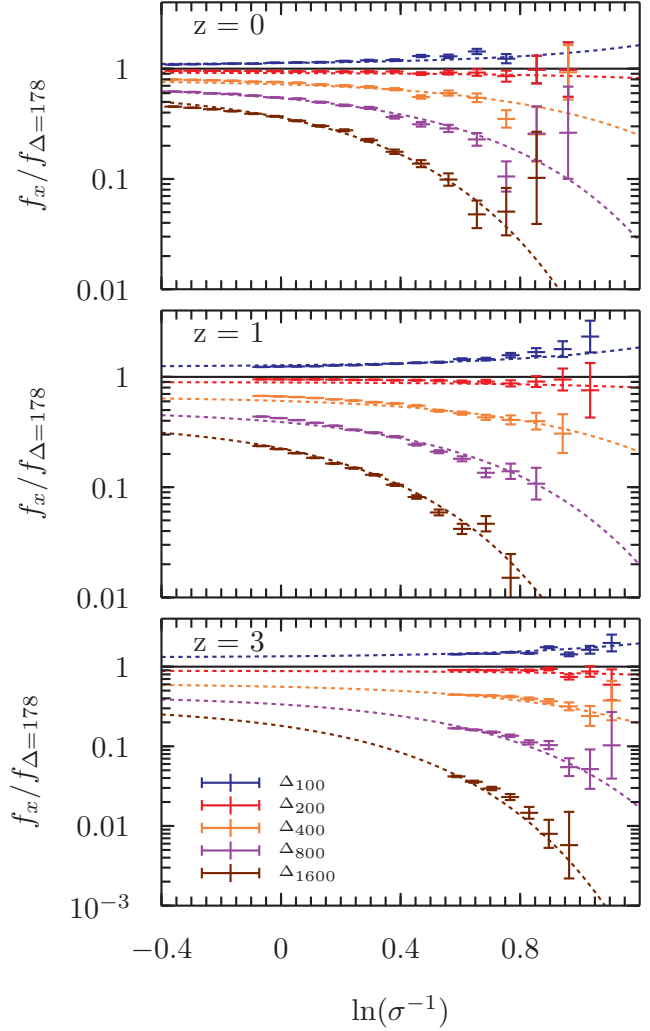
$$f_{\Delta} = \Gamma(\Delta, \sigma, z) f_{\Delta=178} \quad (17)$$

We find that the following form for  $\Gamma$  is suitable for describing our data:

$$\Gamma(\Delta, \sigma, z) = C(\Delta) \left( \frac{\Delta}{178} \right)^{d(z)} \exp \left[ p \left( 1 - \frac{\Delta}{178} \right) / \sigma^q \right] \quad (18)$$

Where:

$$\begin{aligned} C(\Delta) &= 0.947 \exp \left[ 0.023 \left( \frac{\Delta}{178} - 1 \right) \right] \\ d(z) &= -0.456 \Omega_m(z) - 0.139 \\ p &= 0.072 \\ q &= 2.130 \end{aligned} \quad (19)$$



**Figure 11.** The effect of the choice of the overdensity criterion,  $\Delta$ , on the mass function. For  $z = 0, 1, 3$  (top panel to bottom respectively) the ratio of mass functions with a variety of choices for  $\Delta$  are shown relative to the  $\Delta_{178}$  function. All data has been calculated from the  $1h^{-1}$  Gpc simulation. Fitted curves, described in the text, are shown as dashed lines.

The redshift-dependence of the fit is based solely on  $\Omega_m(z)$ , via the  $d(z)$  parameter.

## 5 SUMMARY AND DISCUSSION

### 5.1 Fitted Functions

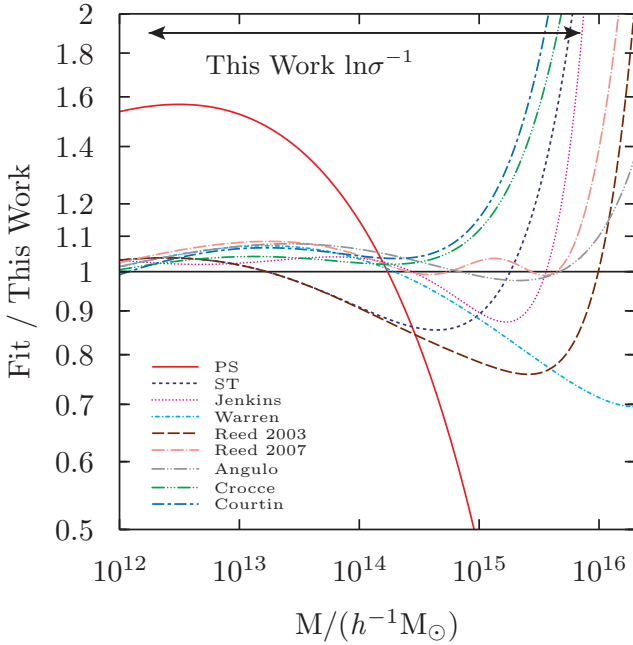
We have provided a number of fitted mass functions based on our data. These are summarised in Table 2.

### 5.2 Comparison to Existing Fits

In Figure 12 we show our universal FOF mass function at  $z = 0$  relative to a number of fits available in the literature, including the widely-used Press-Schechter (PS) (Press & Schechter 1974), Sheth-Tormen (ST) (Sheth & Tormen 2002) and Jenkins et al. (Jenkins et al. 2001) fits, as well as a number of more recent fits (Warren et al. 2006; Reed et al. 2003, 2007; Crocce et al. 2010; Courtin et al. 2011; Angulo et al. 2012). We note that the fits by

**Table 2.** Mass function fitting parameters. For details on the redshift evolution of these parameters see sections 4.5.1 and 4.5.2

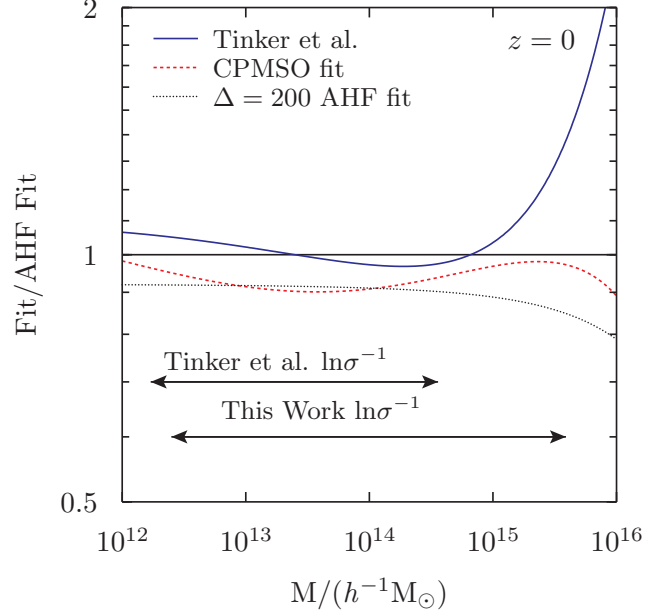
	FOF <i>Uni.</i>	CPMSO	CPMSO +AHF	AHF	AHF <i>EoR</i>
	$z = all$	$z = 0$	$z = 0$	$z = 0$	$z = 6+$
A	0.282	0.287	0.316	0.194	0.563
$\alpha$	1.406	1.478	1.478	2.267	0.874
$\beta$	2.163	2.234	2.234	1.805	3.810
$\gamma$	1.210	1.318	1.318	1.287	1.453
$z$ dep.?	No	Yes	Yes	No	No

**Figure 12.** Ratios between our universal fit for FOF haloes and a number of FOF fits from the literature for  $z = 0$ .

Reed et al. (2007) and Crocce et al. (2010) are redshift-dependent, while Press & Schechter (1974); Sheth & Tormen (2002) and Courtin et al. (2011) are parameterised using  $\delta_c(z)$ . Aside from the classic, and less precise Press & Schechter (1974) expression, our fit agrees to  $\sim 10\%$  with all the others for most of the mass range, and, for very massive clusters ( $> 10^{15} h^{-1} M_\odot$ ) it matches very closely the recent Angulo et al. (2012) fit based on the Millennium series of simulations.

With the exception of the PS, ST and Reed et al. (2003) results, the largest discrepancy both with respect to our fit and overall scatter is for very large, rare haloes, whose statistics in most simulations is poor. A reliable result in this range requires very large simulation volumes of tens to hundreds of  $\text{Gpc}^3$ , comparable to the volume of the observable universe.

As we discussed above, the only other robust SO-based halo mass function fit is the one provided by Tinker et al. (2008). In Figure 13 we show the ratio between our AHF  $z = 0$  fit and the Tinker et al. (2008) fit at  $z = 0$ . We also show how the CPMSO,

**Figure 13.** Ratios, at  $z = 0$ , between the AHF  $z = 0$  fit (solid black line) versus the Tinker et al. (2008) fit (solid blue line), the redshift parameterised CPMSO fit (dotted red line) and the AHF precise fit adjusted to  $\Delta = 200$  using equation 17 (dotted black line).

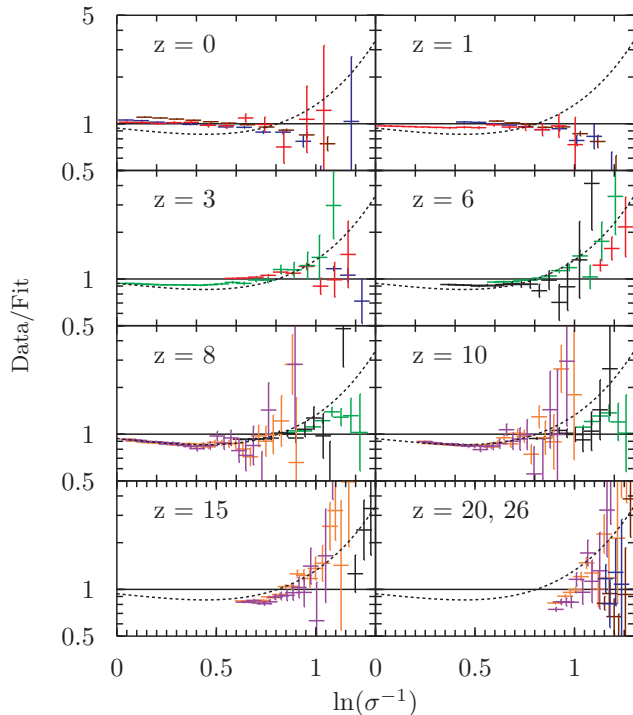
redshift-dependent mass function compares to the AHF  $z = 0$  fit and the ratio between the AHF  $z = 0$  fit and the  $\Delta = 200$  adjusted AHF  $z = 0$  fit (calculated using equation 17). The Tinker et al. (2008) fit is based on data in the range  $-0.6 < \ln \sigma^{-1} \leq 0.4$ , while our fit includes data over the range  $-0.55 < \ln \sigma^{-1} \leq 1.35$  across all redshifts, or specifically  $-0.55 < \ln \sigma^{-1} \leq 1.05$  at  $z = 0$ .

We show an agreement with Tinker et al. (2008) to  $\sim 5\%$  for haloes in the mass range  $10^{12} - 10^{15} h^{-1} M_\odot$ . Past  $\sim 10^{15} h^{-1} M_\odot$  we predict a lower collapsed mass fraction, although this is outside the range of the Tinker et al. (2008) fitting data. The Tinker et al. (2008) fit is based on  $\Delta_{200}$  haloes and we show how our  $\Delta_{200}$  prediction via equation 17 compares to the underlying precision AHF plot. We note that this is slightly less congruent with the Tinker et al. (2008) fit than the  $\Delta_{178}$  prediction, although it is still in agreement to within 10% for most of the mass range in question.

### 5.3 Universality

The results presented here are in line with other recent studies of the halo mass function that have addressed the question of universality, i.e. whether the mass function can be considered invariant over all redshifts or cosmological models when expressed in a suitable form. It is beyond the scope of this paper to discuss invariance under different cosmologies, for work on this see Jenkins et al. (2001); Warren et al. (2006); Tinker et al. (2008) and in particular Courtin et al. (2011) and Bhattacharya et al. (2011). The mass function has been found to depend only weakly on cosmology when couched in terms of FOF haloes with a fixed linking length. More study is required to address the question of how SO halo mass functions differ in varying cosmologies.

We find that the assumption of halo mass function universality across redshift is approximately valid under certain conditions, but is violated in general, in qualitative agreement with the conclusions of other recent studies. Tinker et al. (2008) present an SO mass function that includes redshift-dependent parameters for



**Figure 14.** The redshift evolution of the FOF mass function versus the universal fit. The fit of Sheth & Tormen (2002) is shown as a dashed line for reference.

$z \leq 2.5$ . Similarly Crocce et al. (2010) and Bhattacharya et al. (2011) present FOF mass functions with redshift-dependent parameters for  $z \leq 2$ . Reed et al. (2007) produce a high-redshift mass function that contains a redshift-dependence via the effective slope of the power spectrum at the scale of the halo radius. We observe here that the FOF halo mass function for fixed linking length is close to universal for a very large redshift range (from  $z = 26$  to the present). Jenkins et al. (2001) noted that taking a constant linking length keeps the mass function closer to a universal form, a result corroborated by Lukic et al. (2007) who found that even for high redshifts the mass function of Warren et al. (2006) was suitable for FOF haloes, despite it being calibrated on  $z = 0$  data. However, as we discuss in more detail in § 5.4.1 below, using a fixed linking length for both low and high redshifts is inappropriate and massive FOF haloes often suffer from significant over-linking problems, which results in an over-estimate to the halo mass function in that range. While we derived an universal fit for the FOF halo mass function for fixed linking length of 0.2, there clearly is some, albeit modest, redshift evolution about this fit. We show in Figure 14 the residuals between our FOF universal fit and our data for nine different redshifts. The amplitude of the data relative to the fit drops away markedly for lower  $\ln \sigma^{-1}$  values, to the extent that by  $z \sim 8$  and above the data is around 20% lower than the prediction around  $\ln \sigma^{-1} \sim 0.5$ .

In contrast, the SO-based halo mass functions are clearly not universal over the entire redshift range we consider, which was also previously noted by Tinker et al. (2008) based on data over a smaller range in redshift and  $\sigma$  than our data. Here we presented SO halo mass function fits that are applicable across a large range of redshifts, by explicitly introducing an  $\Omega_m$ - and redshift-dependent parameterisation of the mass function.

## 5.4 Halofinder Comparison

### 5.4.1 Relationship Between SO and FOF Haloes

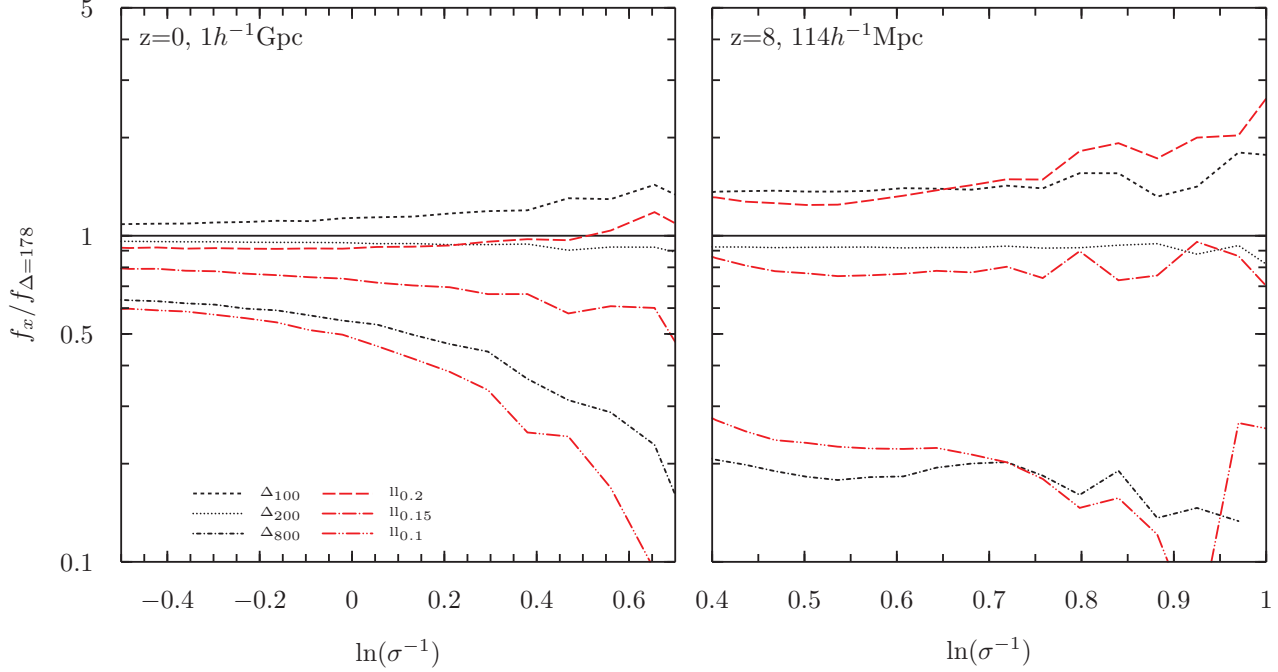
Care must be taken when using an FOF mass function, especially at high redshift, as overlinking of the larger haloes becomes a factor. This occurs when one halo is joined to another by a bridge of particles that all lie within the linking length, thus creating abnormally massive haloes in the simulation catalogue that are not physically meaningful as compact, virialized structures. The differences between an FOF halo and an SO halo, and the respective mass functions derived from haloes found using the two algorithms, are significant. A typical FOF halo is arbitrarily shaped and has been described in the literature as a demarcation of an isodensity contour in real space, with a value of  $\Delta_{\text{FOF}}$  – the overdensity of matter at the halo boundary – that is under some debate. Estimates for  $\Delta_{\text{FOF}}$  include  $\Delta_{\text{FOF}} \approx 2b^{-3}$ , i.e.  $\Delta_{\text{FOF}} \approx 250$  for the standard choice of  $b = 0.2$ , (Navarro et al. 1997);  $\Delta_{\text{FOF}} \approx 0.48b^{-3}$ , i.e.  $\Delta_{\text{FOF}} \approx 60$  (Lacey & Cole 1994; Summers et al. 1995; Audit et al. 1998);  $\Delta_{\text{FOF}} \approx 74$  (Warren et al. 2006); and  $\Delta_{\text{FOF}} \approx 81.62$  More et al. (2011), all for  $b = 0.2$ . In contrast the SO algorithm produces haloes that are by definition spherical and that adhere to a strict overdensity criterion, e.g.  $\Delta = 178$  times the background matter density. This overdensity refers to the average density of matter contained within the halo. Haloes are generally very centrally-concentrated, thus the overdensity of matter at the boundary of an SO halo is much lower than the mean overdensity and the relation between the two is dependent on the profile of the halo in question. More et al. (2011) note that for a singular isothermal sphere (SIS) density profile,  $\rho(r) \propto r^{-2}$ , an overdensity at a halo’s boundary of  $\Delta_{\text{edge}} \sim 60$  corresponds to an enclosed overdensity of  $\Delta_{\text{halo}} \sim 180$ . This loosely links the value of  $b$  in the FOF algorithm to  $\Delta$  in the SO algorithm –by assuming that the FOF halo is spherical– but the simplified SIS profile does not reproduce simulated dark matter haloes well. More appropriate profiles include the TIS profile (Shapiro et al. 1999; Iliev & Shapiro 2001) and the NFW profile. More et al. (2011) used the latter, sampling NFW haloes with random particle realisations and show that their value of  $\Delta_{\text{FOF}} \approx 81.62$  corresponds to a range of  $\Delta_{\text{halo}}$  between  $\approx 200$ –600 depending on the concentration parameter of the NFW haloes. Lukic et al. (2009) use a similar approach and were able to recover, to 5% accuracy, an SO mass function from FOF haloes by individually relating the FOF halo masses to their SO counterparts. The translation between the two was based on the concentration parameter of the haloes’ NFW profiles and three empirically fitted parameters.

### 5.4.2 Choice of Linking Length and Overdensity Criterion

Given these analyses it is not surprising to observe a marked difference in the physical structure of FOF and SO haloes. Figure 15 shows a large halo from our  $20 h^{-1}\text{Mpc}$  simulation at  $z = 8$  captured by the CPMSO and the FOF halo finders, respectively. We show the FOF halo found with linking lengths of 0.2, 0.15, and 0.1 in grey, blue and red respectively. This is the second largest halo in our volume and it illustrates particularly well the differences between the two algorithms. For this halo at this redshift a linking length of 0.2 is far too aggressive and significant overlinking has occurred.

In Figure 16 we compare the halo mass functions based on FOF haloes with linking lengths of 0.2, 0.15, 0.1 and SO haloes with overdensity choices of 100, 178, 200 and 800, at  $z = 0$  from the  $1 h^{-1}\text{Gpc}$  box (left-hand panel) and at  $z = 8$  from the





**Figure 16.** Comparison of mass functions from the  $1h^{-1}\text{Gpc}$  simulation at  $z = 0$  and the  $114h^{-1}\text{Mpc}$  simulation at  $z = 8$  using AHF with a variable overdensity and FOF with a variable linking length. All ratios are plotted on a base of the  $\Delta = 178$  results.

$114h^{-1}\text{Mpc}$  box (right-hand panel). The  $z = 0$  result shows that, for a suitable range of  $\sigma$ , an overdensity of 178 is comparable to a linking length of 0.2 to within 10%. However, for a 0.2 linking length there appears to be a trend towards lower overdensities for the higher mass haloes. This is likely due to the increasing influence of overlinking on the masses of the larger FOF haloes. The  $z = 8$  results show that a linking length of 0.2 is not at all complimentary to an overdensity choice of 178 in the EdS regime of structure growth. In fact, we see that an overdensity of 100 is more consistent with a linking length of 0.2 at  $z = 8$ . The equivalent linking length for an overdensity of 178 lies between 0.2 and 0.15 and much closer to 0.15 than to 0.2, and a linking length of 0.1 roughly corresponds to overdensity of 800. Given these results we expect there to be an evolution in the relationship between overdensity and linking length. The empirical results of Courtin et al. (2011) and More et al. (2011) both contain a redshift dependence (the former via  $\Delta(z)$ , and the latter via the concentration parameter,  $c(z)$ ). Whilst the More et al. (2011) study looked at haloes up to  $z = 2.5$  further study needs to be undertaken to investigate the suitability of existing relations between linking length and overdensity at higher redshifts.

#### 5.4.3 SO vs. FOF Mass Functions

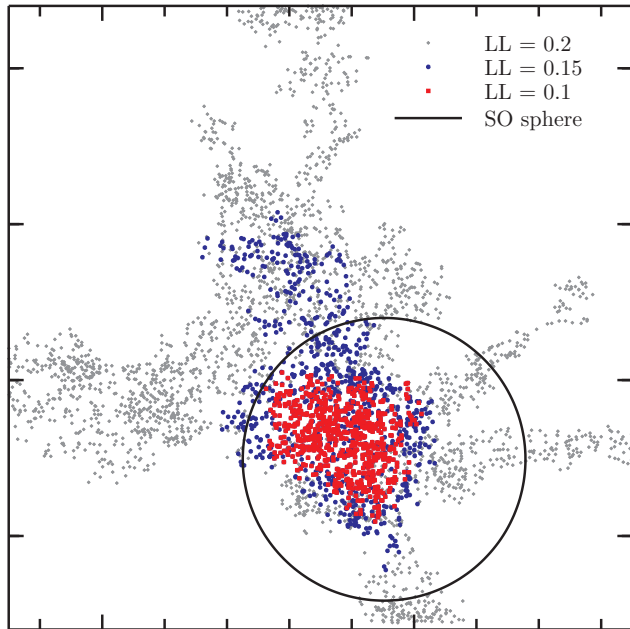
The differences we observed above between haloes found using the two approaches filter through to secondary results derived from halo catalogues, including the mass function. Figure 8 shows the systematic difference between the AHF ( $\Delta = 178$ ) and the FOF ( $b = 0.2$ ) mass functions over a wide range of redshifts. There is a close similarity between the two mass functions at  $z = 0$ , especially for lower mass haloes, with the collapsed fraction of mass in higher-mass haloes greater in the FOF case. Gradually, as we move to higher redshifts, we see the amplitude of the SO mass function fall to around 80% of the FOF mass function at lower

masses, with a much more marked decrease at higher masses. Past redshifts of around  $z \sim 6$  the high mass tail of the SO mass function is around 50% lower than that of the FOF mass function. The causes of the difference between the two mass functions can be summarised as contributions from (1) the relationship between the masses of a given SO halo and its FOF counterpart, which depend on the choices of  $b$  and  $\Delta$  and also on the concentration parameter of the haloes; (2) the amount of overlinking of FOF haloes; (3) the relative mass difference in the two halo types that arises from pseudo mass evolution, as discussed in § 3.4; and (4) other systematic effects including the SO algorithm not correctly interpreting the properties of non-spherical haloes and both algorithms failing to reliably describe merging systems. Lukic et al. (2009) and More et al. (2011) have addressed the first factor in detail, and Davis et al. (1985); Ma & Bertschinger (1995); Cole & Lacey (1996); Lukic et al. (2009) have investigated the second and, to some extent, the fourth factors. The third factor has not been studied in detail and remains a topic for future study.

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**Figure 15.** Image of a large halo in the  $20 h^{-1}\text{Mpc}$  box at  $z = 8$ . The circle represents the extent of the  $\Delta_{178}$  cutoff used in CPMSO. The  $z$  direction has been projected onto the  $x$ - $y$  plane. The CPMSO halo mass is  $3.1 \times 10^{10} h^{-1} M_{\odot}$  and it contains 9.3 million particles. The dots represent aggregations of at least 20 particles found in the FOF version of the same halo. Grey shows the halo captured with a linking length of 0.2, blue 0.15 and red 0.1. The masses (particle counts) are  $4.8 \times 10^{10} h^{-1} M_{\odot}$  (13.1 million),  $3.7 \times 10^{10} h^{-1} M_{\odot}$  (8.9 million) and  $2.1 \times 10^{10} h^{-1} M_{\odot}$  (5.8 million) for  $b = 0.2, 0.15, 0.1$  respectively.

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